

Entry Task:

Complete the square on each of these:

$$x^2 + 10x = \overset{\text{half} \rightarrow 5 \downarrow \text{square}}{x^2 + 10x + 25 - 25} = (x+5)^2 - 25$$

So $x^2 + 10x = (x+5)^2 - 25$

$$4x^2 + 4x + 3 = 4 \left(x^2 + x + \frac{3}{4} \right) = 4 \left(\underbrace{x^2 + x + \frac{1}{4}}_{(x+\frac{1}{2})^2} - \frac{1}{4} + \frac{3}{4} \right) = 4 \left((x+\frac{1}{2})^2 + \frac{1}{2} \right)$$

So $4x^2 + 4x + 3 = 4 \left((x+\frac{1}{2})^2 + \frac{1}{2} \right)$ $\overset{\text{half} \rightarrow 4 \downarrow \text{squared}}$

$$\begin{aligned} 14 - 8x - x^2 &= 14 - (x^2 + 8x) = 14 - (x^2 + 8x + 16 - 16) \\ &= 14 - ((x+4)^2 - 16) \\ &= 14 - (x+4)^2 + 16 \\ &= 30 - (x+4)^2 \end{aligned}$$

So $14 - 8x - x^2 = 30 - (x+4)^2$

7.3 Trigonometric Substitution Summary

CASE	SUBSTITUTION
$\sqrt{a^2 - u^2}$	$u = a \sin(\theta)$ $du = a \cos(\theta) d\theta$ $\sqrt{a^2 - a^2 \sin^2(\theta)} = a \cos(\theta)$
$\sqrt{a^2 + u^2}$	$u = a \tan(\theta)$ $du = a \sec^2(\theta) d\theta$ $\sqrt{a^2 + a^2 \tan^2(\theta)} = a \sec(\theta)$
$\sqrt{u^2 - a^2}$	$u = a \sec(\theta)$ $du = a \sec(\theta) \tan(\theta) d\theta$ $\sqrt{a^2 \sec^2(\theta) - a^2} = a \tan(\theta)$

1. Trig Sub
2. Trig Integral (use 7.2 methods)
3. Triangle Trick

if you encounter a 'middle term'

$$\sqrt{ax^2 + bx + c}$$

Complete the square!

$$\begin{aligned}
 34 - 6x + x^2 &= 34 + x^2 - 6x \\
 &= 34 + x^2 - 6x + 9 - 9 \\
 &= 25 + (x-3)^2
 \end{aligned}$$

Check!!!!

$$25 + x^2 - 6x + 9 \checkmark$$

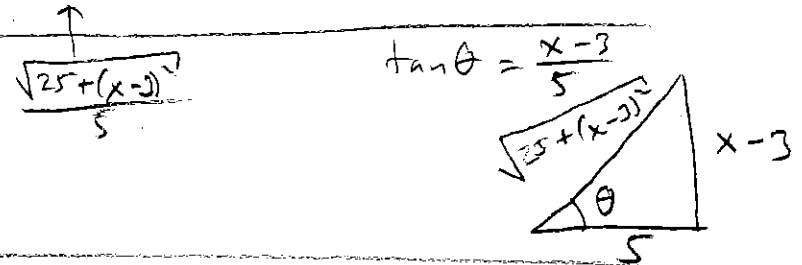
Full Example:

$$\int \frac{x}{\sqrt{34 - 6x + x^2}} dx$$

1. Complete the square.
2. Do a trig sub problem.

$$\begin{aligned}
 &\int \frac{x}{\sqrt{25 + (x-3)^2}} dx && x-3 = 5 \tan(\theta) \\
 & && x = 5 \tan(\theta) + 3 \\
 & && dx = 5 \sec^2(\theta) d\theta \\
 &= \int \frac{(5 \tan(\theta) + 3) 5 \sec^2(\theta) d\theta}{5 \sec \theta}
 \end{aligned}$$

$$\begin{aligned}
 &= \int (5 \tan \theta + 3) \sec \theta d\theta \\
 &= \int 5 \sec \theta \tan \theta + 3 \sec \theta d\theta \\
 &= 5 \sec \theta + 3 \ln |\sec \theta + \tan \theta| + C
 \end{aligned}$$



$$= \boxed{\sqrt{25 + (x-3)^2} + 3 \ln \left| \frac{\sqrt{25 + (x-3)^2}}{5} + \frac{x-3}{5} \right| + C}$$

7.4 Partial Fractions

Motivation: We will learn to break-up fractions like:

$$\frac{x^3 + 4x - 4}{x^2(x^2 + 4)} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2 + 4}$$

Entry Task 1: If I tell you the above relationship is true, then integrate

$$\begin{aligned} & \int \frac{x^3 + 4x - 4}{x^2(x^2 + 4)} dx \\ &= \int \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2 + 4} dx \\ &= \ln|x| + \frac{1}{x} + \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C \end{aligned}$$

NOTE: $\int \frac{1}{x^2 + 4} dx = \frac{1}{4} \int \frac{1}{\frac{x^2}{4} + 1} dx$

$$= \frac{1}{4} \int \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx \quad \begin{array}{l} t = \frac{1}{2}x \\ dt = \frac{1}{2}dx \\ 2dt = dx \end{array}$$

$$= \frac{1}{4} \int \frac{1}{t^2 + 1} 2dt$$

$$= \frac{1}{2} \tan^{-1}(t) + C$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

Entry Task 2: Do you know **long-division**?

Divide in order to fill in the question marks:

$$\frac{576}{11} = ? + \frac{?}{11}$$

$$\begin{array}{r} 52 \\ \hline 11 \overline{) 576} \\ \underline{-55} \\ 26 \\ \underline{-22} \\ 4 \end{array}$$

$$\Rightarrow \frac{576}{11} = 52 + \frac{4}{11}$$

Partial Fraction Decomposition

Step 0: Is the fraction *reduced*?

reduced - highest power on top smaller than the highest power on bottom.

If yes, move to step 1.

If not, divide, then move to step 1.

Example:

$$\int \frac{x^2 + x}{x + 3} dx$$

HIGHEST NUM. POWER = 2

HIGHEST DEN. POWER = 1

$2 \geq 1 \Rightarrow$ DIVIDE!

$$= \int x - 2 + \frac{6}{x + 3} dx$$

$$= \boxed{\frac{1}{2}x^2 - 2x + 6 \ln|x + 3| + C}$$

CHECK $\frac{d}{dx} \rightarrow x - 2 + \frac{6}{x + 3} \checkmark$

$$\begin{array}{r} x - 2 \\ x + 3 \overline{) x^2 + x} \\ \underline{-(x^2 + 3x)} \\ -2x \\ \underline{-(-2x - 6)} \\ 6 \end{array}$$

$$\Rightarrow \frac{x^2 + x}{x + 3} = x - 2 + \frac{6}{x + 3}$$

CHECK $\frac{(x - 2)(x + 3)}{x + 3} + \frac{6}{x + 3}$

$$\frac{x^2 + x - 6 + 6}{x + 3} \checkmark$$

Partial Fractions Method Summary

Step 0: Reduce (if needed), see last page.

Step 1: Factor Denominator.

Write out decomposition below:

i) *Distinct Linear:*

$$\frac{x^2 - 3}{x(x - 1)(x + 4)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 4}$$

ii) *Repeated Linear:*

$$\frac{5 + 2x}{(x + 3)(x - 2)^3} = \frac{A}{x + 3} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2} + \frac{D}{(x - 2)^3}$$

iii) *Irreducible Quadratic:*

$$\frac{4x}{(x + 1)(x^2 + 9)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 9}$$

Step 2: Solve for A, B, C

Step 3: Integrate

All the integrals in this section look like these:

$$\int \frac{1}{2x + 5} dx = \frac{1}{2} \ln|2x + 5| + C$$

$$\int \frac{1}{(x - 4)^2} dx = -\frac{1}{x - 4} + C$$

$$\int \frac{1}{(x + 7)^3} dx = -\frac{1}{2} \frac{1}{(x + 7)^2} + C$$

$$\int \frac{1}{x^2 + 9} dx = \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + C$$

$$\int \frac{x}{x^2 + 9} dx = \frac{1}{2} \ln|x^2 + 9| + C$$

The method uses algebra to rewrite **any** rational function as a sum of the integrals like those above.

Example:

$$\int \frac{x+1}{x^2-4} dx$$

TOP POWER = 1

BOT POWER = 2

$1 < 2 \Rightarrow$

REDUCED ✓

DON'T NEED TO DIVIDE

Factor!

$$\int \frac{1/4}{x+2} + \frac{3/4}{x-2} dx$$

$$\frac{1}{4} \ln|x+2| + \frac{3}{4} \ln|x-2| + C$$

$$\frac{x+1}{(x+2)(x-2)} \stackrel{?}{=} \frac{A}{x+2} + \frac{B}{x-2}$$

$$\Rightarrow x+1 = A(x-2) + B(x+2)$$

$$\Rightarrow x+1 = Ax - 2A + Bx + 2B$$

$$\underbrace{x+1}_{\substack{\uparrow \\ \text{WANT ALWAYS THE SAME!}}} = \underbrace{(A+B)x + (-2A+2B)}$$

MATCH COEFFICIENTS!

$$A+B=1 \Rightarrow B=1-A$$

$$\text{and } -2A+2B=1 \Rightarrow -2A+2(1-A)=1$$

$$\Rightarrow -4A=-1 \Rightarrow A=1/4$$

$$B=1-A=3/4$$

SHORTCUT

$$\text{LET } x=2 \Rightarrow 3 = A(2-2) + B(2+2)$$

$$\Rightarrow B=3/4$$

$$x=-2 \Rightarrow -1 = A(-2-2) + B(-2+2)$$

$$\Rightarrow A=1/4$$

Example:

$$\int \frac{x+1}{x^3+3x^2} dx$$

REDUCED ✓
FACTOR DENOM

$$\frac{x+1}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

↑ REPEATED ↑

$$\int \frac{x+1}{x^2(x+3)} dx$$

$$\Rightarrow x+1 = Ax(x+3) + B(x+3) + Cx^2$$

$$= \int \frac{2/9}{x} + \frac{1/3}{x^2} + \frac{-2/9}{x+3} dx$$

$$= \frac{2}{9} \ln|x| - \frac{1}{3} \frac{1}{x} - \frac{2}{9} \ln|x+3| + C$$

→ $x=0 \Rightarrow 1 = A(0) + B(3) + C(9)$
 $\Rightarrow \boxed{B = 1/3}$

→ $x=-3 \Rightarrow -2 = A(0) + B(0) + C(9)$
 $\Rightarrow \boxed{C = -2/9}$

↓
 $x+1 = Ax^2 + 3Ax + Bx + 3B + Cx^2$
 $x+1 = (A+C)x^2 + (3A+B)x + 3B$

↓
 $A+C=0 \Rightarrow A=-C$
 $3A+B=1$
 $3B=1$

$$\boxed{A = -(-2/9) = 2/9}$$

ASIDE:

$$\frac{2}{9} (\ln|x| - \ln|x+3|) = \frac{2}{9} \ln \left| \frac{x}{x+3} \right|$$

Example:

$$\int \frac{x^2 - x + 6}{x^3 + 3x} dx$$

REDUCED ✓
 FACTOR
 DENOM

$$\frac{x^2 - x + 6}{x(x^2 + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}$$

$$\int \frac{x^2 - x + 6}{x(x^2 + 3)} dx$$

$$\Rightarrow x^2 - x + 6 = A(x^2 + 3) + (Bx + C)x$$

$$= \int \frac{2}{x} + \frac{-x - 1}{x^2 + 3} dx$$

→ SPLIT UP!

$$\begin{aligned} \hookrightarrow x=0 &\Rightarrow 6 = A(3) + (B(0) + C)(0) \\ &\Rightarrow A = 2 \end{aligned}$$

$$= \int \frac{2}{x} dx - \int \frac{x}{x^2 + 3} - \int \frac{1}{x^2 + 3} dx$$

$$x^2 - x + 6 = Ax^2 + 3A + Bx^2 + Cx$$

$$\Rightarrow A + B = 1 \Rightarrow B = 1 - A = -1$$

$$= 2 \ln|x| - \int \frac{x}{u} \frac{1}{2x} du - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$\begin{aligned} C &= -1 \\ 3A &= 6 \quad \checkmark \end{aligned}$$

$$= 2 \ln|x| - \frac{1}{2} \ln|u| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$u = x^2 + 3$$

$$du = 2x dx$$

$$\frac{1}{2x} du = dx$$

$$\boxed{2 \ln|x| - \frac{1}{2} \ln|x^2 + 3| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$$

Example:

$$\int \frac{x}{x^2 + 4x + 5} dx$$

REDUCED ✓

FACTOR DENOM???

$$x^2 + 4x + 5 = 0 \text{ HAS}$$

NO REAL SOL'NS

$$x = \frac{-4 \pm \sqrt{16 - 4(5)}}{2}$$

4 under radical

IRREDUCIBLE

COMPLETE THE SQUARE!

$$x^2 + 4x + 4 - 4 + 5 = (x+2)^2 + 1$$

HALF 2 SQUARES

to simplify

$$\int \frac{x}{(x+2)^2 + 1} dx$$

$$t = x+2 \rightarrow x = t-2$$

$$dt = dx$$

$$\int \frac{t-2}{t^2+1} dt \text{ NOW SPLIT UP!}$$

RE
AST
AGE

$$\int \frac{t}{t^2+1} dt - \int \frac{2}{t^2+1} dt$$

$$= \frac{1}{2} \ln(t^2+1) - 2 \tan^{-1}(t) + C$$

$$= \frac{1}{2} \ln((x+2)^2+1) - 2 \tan^{-1}(x+2) + C$$

How to integrate

A. Look for simplifications/substitutions

B. Products/Logs/Inverse Trig → BY PARTS

Sin/Cos/Tan/Sec combos → TRIG

Quadratic (under a radical) → TRIG SUB

Rational Function → PART. FRAC.

C. If nothing seems to work, substitution.

(u = inside, u = $\sqrt{\quad}$, u = trig, u = e^x)

Examples of substitution:

1. $\int e^{\sqrt{x}} dx$

$$t = \sqrt{x} \Rightarrow t^2 = x \\ 2t dt = dx$$

$$\int e^t 2t dt$$

$$\int 2te^t dt$$

Now BY PARTS!

2. $\int \frac{3}{x - 2\sqrt{x}} dx$

$$\int \frac{3}{t^2 - 2t} 2t dt$$

$$= \int \frac{6t}{t^2 - 2t} dt$$

$$t = \sqrt{x} \Rightarrow t^2 = x \\ 2t dt = dx$$

Now FACTOR DENOMINATOR!

3. $\int \frac{\cos(x)}{4 - \sin^2(x)} dx$

$$\int \frac{1}{4 - t^2} dt$$

$$t = \sin(x) \\ dt = \cos(x) dx$$

Now FACTOR DENOMINATOR!

4. $\int e^x \cos(e^x) \sin^3(e^x) dx$

$$\int \cos(t) \sin^3(t) dt$$

$$t = e^x \\ dt = e^x dx$$

Now 'pull-out'
 $\cos(t)$ AND $u = \sin(t)$!

How would you start these?

$$1. \int \tan^3(x) \sec(x) dx = \int \tan^2(x) \sec(x) \tan(x) dx \quad u = \sec(x)$$

$$\text{TRIG!} = \int (\sec^2(x) - 1) \sec(x) \tan(x) dx \quad du = \sec(x) \tan(x) dx$$

$$2. \int x^2 \ln(x) dx = \frac{1}{3} x^3 \ln(x) - \int \frac{1}{3} x^2 dx$$

$$\text{BY PARTS!} = \frac{1}{3} x^3 \ln(x) - \frac{1}{9} x^3 + C \quad u = \ln(x) \quad dv = x^2 dx$$

$$du = \frac{1}{x} dx \quad v = \frac{1}{3} x^3$$

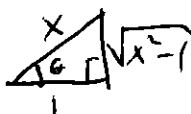
$$3. \int x \sqrt{5-x^2} dx = \int x \sqrt{u} \frac{1}{-2x} du = -\frac{1}{2} \int u^{3/2} du + C \quad u = 5-x^2 \quad \text{EASIER}$$

$$\text{SUB. (OR TRIG SUB)} = -\frac{1}{3} (5-x^2)^{3/2} + C \quad du = -2x dx \quad \text{OR} \quad x = \sqrt{5} \sin(\theta)$$

$$-\frac{1}{2x} du = dx$$

$$4. \int \frac{\sqrt{x^2-1}}{x^2} dx = \int \frac{\sqrt{\sec^2\theta-1}}{\sec^2\theta} \sec\theta \tan\theta d\theta = \int \frac{\tan^2\theta}{\sec\theta} d\theta \quad \text{OR}$$

$$\text{TRIG SUB} \quad \int \frac{\sec^2\theta-1}{\sec\theta} d\theta = \int \frac{\sin^2\theta}{\cos\theta} d\theta \quad x = \sec\theta$$

$$dx = \sec\theta \tan\theta d\theta$$


$$5. \int \frac{x^2+1}{x^2-2x-3} dx = \int 1 + \frac{2x+4}{x^2-2x-3} dx$$

$$\text{DIVIDE, THEN FACTOR DENOM} \quad \frac{2x+4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

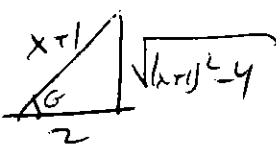
$$x^2-2x-3 \overline{) \frac{1}{x^2+1} } \quad \frac{-(x^2-2x-3)}{2x+4}$$

$$6. \int x \tan^{-1}(x) dx = \frac{1}{2} x^2 \tan^{-1}(x) - \frac{1}{2} \int \frac{x^2}{x^2+1} dx$$

$$\text{BY PARTS} \quad \triangle \text{ DIVIDE!!} \quad u = \tan^{-1}(x) \quad dv = x dx$$

$$du = \frac{1}{x^2+1} dx \quad v = \frac{1}{2} x^2$$

$$7. \int \frac{dx}{\sqrt{4x^2+8x-12}} dx \quad \sqrt{4(x^2+2x+1-3)} = 2\sqrt{(x+1)^2-4}$$

$$\text{COMPLETE SQUARE \& TRIG SUB!} \quad \Rightarrow \int \frac{1}{2\sqrt{4\sec^2\theta-4}} 2\sec\theta \tan\theta d\theta \quad x+1 = 2\sec\theta$$


$$= \frac{1}{2} \int \frac{1}{2\tan\theta} 2\sec\theta \tan\theta d\theta = \frac{1}{2} \ln|\sec\theta + \tan\theta| + C$$